

The Fubini–Bruhat orders

Sara Billey

University of Washington

Based on joint work with Stark Ryan

Fubini words are generalized permutations, allowing for repeated letters, and they are in one-to-one correspondence with ordered set partitions. Brendan Pawlowski and Brendon Rhoades extended permutation matrices to pattern matrices for Fubini words. Under a lower triangular action, these pattern matrices produce cells in projective space, specifically $(\mathbb{P}^{k-1})^n$. The containment of the cell closures in the Zariski topology gives rise to a poset which generalizes the Bruhat order for Schubert cells/varieties indexed by permutations. Unlike Bruhat order, containment is not equivalent to intersection of a cell with the closure of another cell. This allows for a refinement of the poset. It is additionally possible to define a weaker order, giving rise to a subposet still containing all the elements. We call these orders, in order of decreasing strength, the espresso, medium roast, and decaf Fubini-Bruhat orders. The espresso and medium roast orders are not ranked in general. The decaf order is ranked by codimension of the corresponding cells. In fact, the decaf order has rank generating function given by a well-known q -analog of the Stirling numbers of the second kind. We give increasingly smaller sets of equations describing the cell closures, which lead to several different combinatorial descriptions for the relations in all three orders. We also describe a few classes of covering relations in each of the orders. We will conclude with several interesting open problems in this area.