Bender–Knuth Billiards in Coxeter Groups

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Let (W, S) be a Coxeter system, and write $S = \{s_i : i \in I\}$, where I is a finite index set. Consider a nonempty finite convex subset $\mathscr{L}$ of W. If W is a symmetric group, then $\mathscr{L}$ is the set of linear extensions of a poset, and there are important *Bender–Knuth involutions* $\mathrm{BK}_i \colon \mathscr{L} \to \mathscr{L}$ indexed by elements of I. For arbitrary W and for each $i \in I$, we introduce an operator $\tau_i \colon W \to W$ that we call a *noninvertible Bender–Knuth toggle*; this operator restricts to an involution on $\mathscr{L}$ that coincides with BK_i when W is a symmetric group. Given an ordering $i_1, \ldots, i_n$ of I and a starting element u_0 of W, we can repeatedly apply the toggles in the order $\tau_{i_1}, \ldots, \tau_{i_n}, \tau_{i_1}, \ldots, \tau_{i_n}, \ldots$; this produces a sequence of elements of W that can be viewed in terms of a beam of light that bounces around in an arrangement of transparent windows and one-way mirrors. Our central questions concern whether or not the beam of light eventually ends up in $\mathscr{L}$. We will discuss several situations where this occurs and several situations where it does not.