q,t-Catalan $\times$ Hall–Littlewood $\times$ Rogers–Ramanujan?

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Based on joint work with Ruofan Jiang

It is conjectured (though a variant is well-known) that the Hilbert schemes of points on the planar singular curve $y^m = x^n$ recover the rational q,t-Catalan number (at least when $\gcd(m,n) = 1$) defined as $\sum q^{\text{area}} t^{\text{dinv}}$ summed over $m \times n$ Dyck paths [1, 3]. In view of this correspondence, the q,t-symmetry corresponds to a functional equation resulting from the Serre duality in algebraic geometry.

We propose a generalization of this algebro-geometric model that surprisingly yields a marriage of all three topics in the title [2]. Fix an integer $d \geq 1$, consider the Quot schemes parametrizing finite-length quotients of a rank d vector bundle on the said curve. This includes the Hilbert scheme of points as a $d = 1$ special case. Our results suggest that certain Catalan-like features persist in the high-d generalization: an analogous functional equation holds, and the case $m = 2$, $n \geq 2$ exhibits proven/conjectured positivity patterns. The said family of cases (note that they are the baby cases in the Dyck path theory) yield q,t-polynomials whose formulas surprisingly involve the Hall polynomial, an ingredient absent in the known $d = 1$ case but present whenever $d \geq 2$. Moreover, by specializing and taking a $d \to \infty$ limit (which has an independent number-theoretic motivation from counting commuting matrices), these polynomials give rise to infinite sums that evaluate to infinite products, generalizing the celebrated Rogers–Ramanujan identities in partition theory.

References

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