Off-diagonally symmetric domino tilings of the Aztec diamond

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Consider a finite group G acting on a set of combinatorial objects X. Let H be a subgroup of G. A *symmetry class* is a collection of H-invariant objects of X. In enumerative combinatorics, it is quite challenging to enumerate each symmetry class of X because the structure of each class varies with different subgroups H. These usually require different methods to enumerate them. The symmetry classes of plane partitions and alternating sign matrices have been studied extensively for the past forty years, see for instance [4] and [1, Section 1.2].

Let X be the set of domino tilings of the Aztec diamond of order n and $G = \langle r, t | r^4 = t^2 = (tr)^2 = \mathrm{id} \rangle$ the dihedral group of order 8. It turns out that this group action gives five (nontrivial) symmetry classes listed below. Enumerating the first three symmetry classes has been solved, and they all have nice counting formulas ([6], [2]). For the last two symmetry classes, finding a closed-form formula is remained open.

Subgroup of G	Symmetry Class	Size and Reference
$H_1 = \{\mathrm{id}\}$	Original Aztec diamond	$\mathrm{M}_{H_1}(n) = 2^{n(n+1)/2}$.
$H_2 = \langle r \rangle$	Quarter-turn invariant	$\mathrm{M}_{H_2}(n)$ has a product formula.
$H_3 = \langle r^2 \rangle$	Half-turn invariant	$\mathrm{M}_{H_3}(n)$ has a product formula.
$H_4 = \langle t \rangle$	Diagonally symmetric	$\mathrm{M}_{H_4}(n) = 2, 6, 24, 132, 1048, 11960, \ldots$
$H_5 = \langle r^2, t \rangle$	Diagonally and anti-diagonally symmetric	$\mathrm{M}_{H_5}(n) = 2, 4, 10, 28, 96, 384, 1848, \ldots$

We introduce a new symmetry class of domino tilings of the Aztec diamond, called the *off-diagonal* symmetry class, which is motivated by the off-diagonally symmetric alternating sign matrices introduced by Kuperberg in 2002 [3]. We use the method of non-intersecting lattice paths and a modification of Stembridge's Pfaffian formula [5] for families of non-intersecting lattice paths to enumerate our new symmetry class. The number of off-diagonally symmetric domino tilings of the Aztec diamond can be expressed as a Pfaffian of a matrix whose entries satisfy a nice and simple recurrence relation.

References

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