# Kromatic symmetric functions

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*Based on joint work with Logan Crew and Sophie Spirkl*

Stanley's [3] chromatic symmetric function $\chi(G)$ of a graph $G$ simultaneously encodes information about stable decompositions, the contraction lattice, and acyclic orientations. One of the main open problems is the Stanley–Stembridge conjecture that chromatic symmetric functions of claw-free incomparability graphs are $e$-positive. There have been many exciting developments lately, leading to resolutions of special cases of this conjecture. However, the best general result that we have is Gasharov's weaker theorem [2] that chromatic symmetric functions of claw-free incomparability graphs are Schur-positive.

Schur-positivity often means that your symmetric function stands in for a finite-dimensional complex representation of the symmetric group or the special linear group. Indeed, most of the recent progress on the Stanley–Stembridge conjecture has come from this perspective.

But there's another potential explanation for Schur-positivity. A symmetric function is Schur-positive when it represents the cohomology class of a subvariety of a Grassmannian. We would like such an explanation of Gasharov's theorem: For every claw-free incomparability graph $G$, produce a variety $V$ with $[V] = \chi(G)$.

Well, I don't know how to do that. But we carry out a test of plausibility, introducing a new $K$-theoretic deformation of the chromatic symmetric function and showing that those in the Stanley—Stembridge setting are positive in the basis of symmetric Grothendieck polynomials [1]. We imagine this deformation should encode the structure sheaf class of the variety $V$. Our new invariant differentiates many graphs that are not distinguished by the ordinary chromatic symmetric function, but there remain many mysteries.

## References

[1] L. Crew, O. Pechenik and S. Spirkl. The Kromatic symmetric function: A $K$-theoretic analogue of $X_G$. `arXiv:2301.02177`, 2023: 16 pages.

[2] V. Gasharov. Incomparability graphs of $(3+1)$-free posets are $s$-positive. *Discrete Math.*, 1996: 193–197.

[3] R. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, 1995: 166–194.